

STABILITY OF SQUARE ROOT DOMAINS ASSOCIATED WITH ELLIPTIC SYSTEMS OF PDES ON NONSMOOTH DOMAINS

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ABSTRACT. We discuss stability of square root domains for uniformly elliptic partial differential operators $L_{a,\Omega,\Gamma} = -\nabla \cdot a \nabla$ in $L^2(\Omega)$, with mixed boundary conditions on $\partial\Omega$, with respect to additive perturbations. We consider open, bounded, and connected sets $\Omega \in \mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, that satisfy the interior corkscrew condition and prove stability of square root domains of the operator $L_{a,\Omega,\Gamma}$ with respect to additive potential perturbations $V \in L^p(\Omega) + L^\infty(\Omega)$, $p > n/2$.

Special emphasis is put on the case of uniformly elliptic systems with mixed boundary conditions.

1. INTRODUCTION

The aim of this note is to provide applications of a recently developed abstract approach to the stability of square root domains of non-self-adjoint operators with respect to additive perturbations to elliptic partial differential operators with mixed boundary conditions on a class of open, bounded, connected sets $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, that satisfy the corkscrew condition (and hence go beyond bounded Lipschitz domains).

More precisely, if T_0 is an appropriate non-self-adjoint m -accretive operator in a separable, complex Hilbert space \mathcal{H} , we developed an abstract approach in [18] to determine conditions under which non-self-adjoint additive perturbations W of T_0 yield the stability of square root domains in the form

$$\operatorname{dom}((T_0 + W)^{1/2}) = \operatorname{dom}(T_0^{1/2}). \quad (1.1)$$

In fact, driven by applications to PDEs, we were particularly interested in the following variant of this stability problem for square root domains with respect to additive perturbations: if T_0 is an appropriate non-self-adjoint operator for which it is known that Kato's square root problem in the following abstract form, that is,

$$\operatorname{dom}(T_0^{1/2}) = \operatorname{dom}((T_0^*)^{1/2}) \quad (1.2)$$

is valid, for which non-self-adjoint additive perturbations W of T_0 can one conclude that also

$$\operatorname{dom}((T_0 + W)^{1/2}) = \operatorname{dom}(T_0^{1/2}) = \operatorname{dom}((T_0^*)^{1/2}) = \operatorname{dom}(((T_0 + W)^*)^{1/2}) \quad (1.3)$$

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holds?

Without going into details at this point we note that $T_0 + W$ will be viewed as a form sum of T_0 and W .

Formally speaking, the role of the operator T_0 in \mathcal{H} in this note will be played by $L_{a,\Omega,\Gamma}$, an m -sectorial realization of the uniformly elliptic differential expression in divergence form, $-\nabla \cdot a \nabla$, in $L^2(\Omega)$, with $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$ an open, bounded, connected set that satisfies the corkscrew condition, and the coefficients $a_{j,k}$, $1 \leq j, k \leq n$, assumed to be essentially bounded. Moreover, $L_{a,\Omega,\Gamma}$ is constructed in such a manner via quadratic forms so that it satisfies a Dirichlet boundary condition along the closed (possibly empty) subset $\Gamma \subseteq \partial\Omega$ and a Neumann boundary condition on the remainder of the boundary, $\partial\Omega \setminus \Gamma$. The additive perturbation W of T_0 then is given by a potential term V , that is, by an operator of multiplication in $L^2(\Omega)$ by an element

$$V \in L^p(\Omega) + L^\infty(\Omega) \text{ for some } p > n/2. \quad (1.4)$$

(For simplicity we will not consider the one-dimensional case $n = 1$ in this note as that has been separately discussed in [19].)

In fact, we will go a step further and consider uniformly elliptic systems in $L^2(\Omega)^N$, $N \in \mathbb{N}$, where T_0 in \mathcal{H} is represented by $\mathbf{L}_{a,\Omega,\mathbb{G}}$ in $L^2(\Omega)^N$, the m -sectorial realization of the $N \times N$ matrix-valued differential expression \mathbf{L} which acts as

$$\mathbf{L}u = - \left(\sum_{j,k=1}^n \partial_j \left(\sum_{\beta=1}^N a_{j,k}^{\alpha,\beta} \partial_k u_\beta \right) \right)_{1 \leq \alpha \leq N}, \quad u = (u_1, \dots, u_N), \quad (1.5)$$

with $a_{j,k}^{\alpha,\beta} \in L^\infty(\Omega)$, $1 \leq j, k \leq n$, $1 \leq \alpha, \beta \leq N$. Here \mathbb{G} represents the collection $\mathbb{G} = (\Gamma_1, \Gamma_2, \dots, \Gamma_N)$, with $\Gamma_\alpha \subseteq \partial\Omega$ a closed (possibly empty) subset of $\partial\Omega$, and intuitively, \mathbf{L} acts on vectors $u = (u_1, u_2, \dots, u_N)$, where each component u_α formally satisfies a Dirichlet boundary condition along Γ_α and a Neumann condition along the remainder of the boundary, $\partial\Omega \setminus \Gamma_\alpha$, $1 \leq \alpha \leq N$. The additive perturbation W of T_0 then corresponds to an $N \times N$ matrix-valued operator of multiplication in $L^2(\Omega)^N$ of the form

$$(\mathbf{V}f)_\alpha = \sum_{\beta=1}^N V_{\alpha,\beta} f_\beta, \quad 1 \leq \alpha \leq N, \quad f \in \text{dom}(\mathbf{V}) = \{f \in L^2(\Omega)^N \mid \mathbf{V}f \in L^2(\Omega)^N\}. \quad (1.6)$$

with

$$V_{\alpha,\beta} \in L^p(\Omega) + L^\infty(\Omega) \text{ for some } p > n/2, \quad 1 \leq \alpha, \beta \leq N. \quad (1.7)$$

The considerable amount of literature on Kato's square root problem in the concrete case where T_0 represents a uniformly elliptic differential operator in divergence form $-\nabla \cdot a \nabla$ in $L^2(\Omega)$ with various boundary conditions on $\partial\Omega$, has been reviewed in great detail in [18]. Thus, in this note we now confine ourselves to refer, for instance, in addition to [2], [3], [4], [5], [6], [7], [8], [12], [9], [10], [11], [13], [14], [15], [21], [23], [24], [27], and the references cited in these sources.

The starting point for this note was a recent paper by Egert, Haller-Dintelmann, and Tolksdorf [14] (cf. Theorem 2.5), which permits us to go beyond the class of strongly Lipschitz domains considered in [18] and now consider open, bounded, and connected sets $\Omega \subset \mathbb{R}^n$ that satisfy the interior corkscrew condition. In Section 2 we first consider uniformly elliptic partial differential operators with mixed boundary conditions on Ω , closely following [14], and subsequently study the quadratic forms

associated with $L_{a,\Omega,\Gamma}$ and V . We then prove stability of square root domains of the operator $L_{a,\Omega,\Gamma}$ with respect to additive perturbations $V \in L^p(\Omega) + L^\infty(\Omega)$, $p > n/2$, for this more general class of domains Ω . The extension of these results to elliptic systems governed by (1.5) and perturbed by the matrix-valued potentials in (1.6) then is the content of Section 3.

Finally, we briefly summarize some of the notation used in this paper: Let \mathcal{H} be a separable, complex Hilbert space with scalar product (linear in the second argument) and norm denoted by $(\cdot, \cdot)_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$, respectively. Next, if T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\text{dom}(T)$ denotes the domain of T . The closure of a closable operator S is denoted by \overline{S} . The form sum of two (appropriate) operators T_0 and W is abbreviated by $T_0 +_q W$.

The Banach space of bounded linear operators on a separable complex Hilbert space \mathcal{H} is denoted by $\mathcal{B}(\mathcal{H})$. The notation $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ is used for the continuous embedding of the Banach space \mathcal{X}_1 into the Banach space \mathcal{X}_2 .

If $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ is a bounded set, then $\text{diam}(\Omega) = \sup_{x,y \in \Omega} |x - y|$ denotes the diameter of Ω . We use $m_{\ell,n}$ to denote the ℓ -dimensional Hausdorff measure on \mathbb{R}^n (and hence $m_{n,n}$, also denoted by $|\cdot|$, represents the n -dimensional Lebesgue measure on \mathbb{R}^n). If $x \in \mathbb{R}^n$ and $r > 0$, then $B(x, r)$ denotes the open ball of radius r centered at x . In addition, I_n denotes the $n \times n$ identity matrix in \mathbb{C}^n , and the set of $k \times \ell$ matrices with complex-valued entries is denoted by $\mathbb{C}^{k \times \ell}$. Finally, we abbreviate $L^p(\Omega; d^n x) := L^p(\Omega)$ and $L^p(\Omega, \mathbb{C}^N; d^n x) := L^p(\Omega)^N$, $N \in \mathbb{N}$.

2. ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS WITH MIXED BOUNDARY CONDITIONS

In this section we discuss stability of square root domains for uniformly elliptic partial differential operators $L_{a,\Omega,\Gamma} = -\nabla \cdot a \nabla$ in $L^2(\Omega)$, with mixed boundary conditions on $\partial\Omega$, with respect to additive perturbations in [18], by employing a recent result due to Egert, Haller-Dintelmann, and Tolksdorf [14] (recorded in Theorem 2.5 below). This permits us to go beyond the class of strongly Lipschitz domains considered in [18] and now consider open, bounded, and connected sets $\Omega \subset \mathbb{R}^n$ that satisfy the interior corkscrew condition. We then prove stability of square root domains of the operator $L_{a,\Omega,\Gamma}$ with respect to additive potential perturbations $V \in L^p(\Omega) + L^\infty(\Omega)$, $p > n/2$, for this more general class of domains Ω .

We start with the following definitions:

Definition 2.1. Let $n \in \mathbb{N} \setminus \{1\}$.

(i) A nonempty, bounded, open, and connected set $\Omega \subset \mathbb{R}^n$ is said to satisfy the interior corkscrew condition if there exists a constant $\kappa \in (0, 1)$ with the property that for each $x \in \overline{\Omega}$ and each $r \in (0, \text{diam}(\Omega))$, there exists a point $y \in \overline{B(x, r)}$ such that $\overline{B(y, \kappa r)} \subseteq \Omega$.

(ii) Let Ω be a nonempty, proper, open subset of \mathbb{R}^n . One calls Ω a Lipschitz domain if for every $x_0 \in \partial\Omega$ there exist $r > 0$, a rigid transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and a Lipschitz function $\varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ with the property that

$$T(\Omega \cap B(x_0, r)) = T(B(x_0, r)) \cap \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n > \varphi(x')\}. \quad (2.1)$$

We recall our convention that $m_{\ell,n}$ denotes the ℓ -dimensional Hausdorff measure (for the basics on Hausdorff measure, see, e.g., [17, Ch. 2], [25, Ch. 2]) and hence $m_{n,n} = |\cdot|$ represents n -dimensional Lebesgue measure on \mathbb{R}^n .

The following proposition records some basic results in connection with the interior corkscrew condition.

Proposition 2.2. *Let $n \in \mathbb{N} \setminus \{1\}$ and suppose that $\Omega \subset \mathbb{R}^n$ is nonempty, bounded, open, and connected. Then the following items (i) and (ii) hold:*

(i) *If Ω satisfies the interior corkscrew condition with constant $\kappa \in (0, 1)$, then*

$$\kappa^n r^n \leq |\Omega \cap B(x, r)| \leq r^n, \quad x \in \Omega, \quad 0 < r < \text{diam}(\Omega). \quad (2.2)$$

(ii) *If Ω is a bounded Lipschitz domain, then Ω satisfies the interior corkscrew condition.*

Definition 2.3. *Suppose $n \in \mathbb{N}$ is fixed and $0 < \ell \leq n$. A non-empty Borel set $M \subseteq \mathbb{R}^n$ is an ℓ -set if there exist constants $c_j = c_j(M) > 0$, $j = 1, 2$, for which*

$$c_1 r^\ell \leq m_{\ell,n}(M \cap B(x, r)) \leq c_2 r^\ell, \quad x \in M, \quad 0 < r \leq 1. \quad (2.3)$$

One notes that \overline{M} is an ℓ -set if M is an ℓ -set and $m_{\ell,n}(\overline{M} \setminus M) = 0$ in this case.

Hypothesis 2.4. *Let $n \in \mathbb{N} \setminus \{1\}$.*

(i) *Assume that $\Omega \subset \mathbb{R}^n$ is a nonempty, bounded, open, and connected set satisfying the interior corkscrew condition in Definition 2.1 (i).*

(ii) *Suppose $\Gamma \subset \partial\Omega$ is closed and for every $x \in \overline{\partial\Omega \setminus \Gamma}$, there exists an open neighborhood $U_x \subset \mathbb{R}^n$ and a bi-Lipschitz map $\Phi_x : U_x \rightarrow (-1, 1)^n$ such that*

$$\Phi_x(x) = 0, \quad (2.4)$$

$$\Phi_x(\Omega \cap U_x) = (-1, 1)^{n-1} \times (-1, 0), \quad (2.5)$$

$$\Phi_x(\partial\Omega \cap U_x) = (-1, 1)^{n-1} \times \{0\}. \quad (2.6)$$

(iii) *Suppose $\Gamma = \emptyset$ or Γ is an $(n-1)$ -set.*

(iv) *Assume that $a : \Omega \rightarrow \mathbb{C}^{n \times n}$ is a Lebesgue measurable, matrix-valued function which is essentially bounded and uniformly elliptic, that is, there exist constants $0 < a_1 \leq a_2 < \infty$ such that for a.e. $x \in \Omega$,*

$$a_1 \|\xi\|_{\mathbb{C}^n}^2 \leq \text{Re}[(\xi, a(x)\xi)_{\mathbb{C}^n}] \quad \text{and} \quad |(\xi, a(x)\xi)_{\mathbb{C}^n}| \leq a_2 \|\xi\|_{\mathbb{C}^n} \|\zeta\|_{\mathbb{C}^n}, \quad \xi, \zeta \in \mathbb{C}^n. \quad (2.7)$$

(v) *With $C_\Gamma^\infty(\Omega)$ defined by*

$$C_\Gamma^\infty(\Omega) := \{u|_\Omega \mid u \in C^\infty(\mathbb{R}^n), \text{dist}(\text{supp}(u), \Gamma) > 0\}, \quad (2.8)$$

denote by $W_\Gamma^{1,2}(\Omega)$ the closure of $C_\Gamma^\infty(\Omega)$ in $W^{1,2}(\Omega)$, that is,

$$W_\Gamma^{1,2}(\Omega) = \overline{C_\Gamma^\infty(\Omega)}^{W^{1,2}(\Omega)}, \quad (2.9)$$

and introduce the densely defined, accretive, and closed sesquilinear form in $L^2(\Omega)$,

$$\mathfrak{q}_{a,\Omega,\Gamma}(f, g) = \int_\Omega d^n x ((\nabla f)(x), a(x)(\nabla g)(x))_{\mathbb{C}^n}, \quad f, g \in \text{dom}(\mathfrak{q}_{a,\Omega,\Gamma}) := W_\Gamma^{1,2}(\Omega). \quad (2.10)$$

We denote by $L_{a,\Omega,\Gamma}$ the m -sectorial operator in $L^2(\Omega)$ uniquely associated to $\mathfrak{q}_{a,\Omega,\Gamma}$.

(vi) *Suppose that $V : \Omega \rightarrow \mathbb{C}$ is (Lebesgue) measurable and factored according to*

$$V(x) = u(x)v(x), \quad v(x) = |V(x)|^{1/2}, \quad u(x) = e^{i \arg(V(x))} v(x) \quad \text{for a.e. } x \in \Omega, \quad (2.11)$$

such that

$$W_\Gamma^{1,2}(\Omega) \subseteq \text{dom}(v). \quad (2.12)$$

In the special case where $a(x) = I_n$ for a.e. $x \in \Omega$, with I_n the $n \times n$ identity matrix in \mathbb{C}^n , we simplify notation and write

$$L_{I_n, \Omega, \Gamma} = -\Delta_{\Omega, \Gamma}. \quad (2.13)$$

Note that $-\Delta_{\Omega, \Gamma}$ is self-adjoint and non-negative.

For an example of a bounded, open, and connected set that satisfies the conditions (i)–(iii) of Hypothesis 2.4 and is not Lipschitz, see [14, Figure 1]. One notes that Hypothesis 2.4 (i) permits *inward-pointing* cusps.

Formally speaking, the operator $L_{a, \Omega, \Gamma}$ is of uniform elliptic divergence form $L_{a, \Omega, \Gamma} = -\nabla \cdot a \nabla$, satisfying a Dirichlet boundary condition along Γ and a Neumann (or, natural) boundary condition on the remainder of the boundary, $\partial\Omega \setminus \Gamma$.

The quadratic form \mathfrak{q}_V in $L^2(\Omega)$, uniquely associated with V , is defined by

$$\mathfrak{q}_V(f, g) = (vf, e^{i \arg(V)} vg)_{L^2(\Omega)}, \quad f, g \in \text{dom}(\mathfrak{q}_V) = \text{dom}(v). \quad (2.14)$$

Under appropriate assumptions on V (see Hypotheses 2.6 and 2.7 below), the form sum of $\mathfrak{q}_{a, \Omega, \Gamma}$ and \mathfrak{q}_V will define a sectorial form on $W_\Gamma^{1,2}(\Omega)$ and the operator uniquely associated to $\mathfrak{q}_{a, \Omega, \Gamma} + \mathfrak{q}_V$ will be denoted by $L_{a, \Omega, \Gamma} + \mathfrak{q}_V$ (see also the paragraph following [18, eq. (A.42)]).

The principal aim of this section is to prove stability of square root domains in the form

$$\text{dom}((L_{a, \Omega, \Gamma} + \mathfrak{q}_V)^{1/2}) = \text{dom}(L_{a, \Omega, \Gamma}^{1/2}) = W_\Gamma^{1,2}(\Omega), \quad (2.15)$$

under appropriate (integrability) assumptions on V , thereby extending the recent results on stability of square root domains obtained in [18] to the setting of certain classes of non-Lipschitz domains with mixed boundary conditions as discussed in [14]. As a basic input, we rely on the following result which is Theorem 4.1 in [14].

Theorem 2.5 (Egert–Haller–Dintelmann–Tolksdorf [14]). *Assume items (i)–(v) of Hypothesis 2.4. Then*

$$\text{dom}(L_{a, \Omega, \Gamma}^{1/2}) = \text{dom}((L_{a, \Omega, \Gamma}^*)^{1/2}) = W_\Gamma^{1,2}(\Omega). \quad (2.16)$$

Next, we introduce various hypotheses corresponding to the potential coefficient V .

Hypothesis 2.6. *Let $n \in \mathbb{N} \setminus \{1\}$, assume that $\Omega \subseteq \mathbb{R}^n$ is nonempty and open, and let $V \in L^p(\Omega) + L^\infty(\Omega)$ for some $p > n/2$.*

In addition, we also discuss the critical L^p -index $p = n/2$ for V for $n \geq 3$:

Hypothesis 2.7. *Let $n \in \mathbb{N} \setminus \{1, 2\}$, assume that $\Omega \subseteq \mathbb{R}^n$ is nonempty and open, and let $V \in L^{n/2}(\Omega) + L^\infty(\Omega)$.*

Here $V \in L^q(\Omega) + L^\infty(\Omega)$ means as usual that V permits a decomposition $V = V_q + V_\infty$ with $V_q \in L^q(\Omega)$ for some $q \geq 1$ and $V_\infty \in L^\infty(\Omega)$.

Theorem 2.8. *Assume Hypotheses 2.4 and 2.6. Then the following items (i) and (ii) hold:*

(i) *V is infinitesimally form bounded with respect to $-\Delta_{\Omega, \Gamma}$, and there exist constants $M > 0$ and $\varepsilon_0 > 0$ such that*

$$\begin{aligned} \| |V|^{1/2} f \|_{L^2(\Omega)}^2 &\leq \varepsilon \| (-\Delta_{\Omega, \Gamma})^{1/2} f \|_{L^2(\Omega)}^2 + M \varepsilon^{-n/(2p-n)} \| f \|_{L^2(\Omega)}^2, \\ f &\in W_\Gamma^{1,2}(\Omega), \quad 0 < \varepsilon < \varepsilon_0. \end{aligned} \quad (2.17)$$

(ii) V is infinitesimally form bounded with respect to $L_{a,\Omega,\Gamma}$ and

$$\begin{aligned} \| |V|^{1/2} f \|_{L^2(\Omega)}^2 &\leq \varepsilon \operatorname{Re}[\mathfrak{q}_{a,\Omega,\Gamma}(f, f)] + M a_1^{-n/(2p-n)} \varepsilon^{-n/(2p-n)} \|f\|_{L^2(\Omega)}^2, \\ f &\in W_\Gamma^{1,2}(\Omega), \quad 0 < \varepsilon < a_1^{-1} \varepsilon_0. \end{aligned} \quad (2.18)$$

The form sum $L_{a,\Omega,\Gamma} +_{\mathfrak{q}} V$ is an m -sectorial operator which satisfies

$$\begin{aligned} \operatorname{dom}((L_{a,\Omega,\Gamma} +_{\mathfrak{q}} V)^{1/2}) &= \operatorname{dom}(((L_{a,\Omega,\Gamma} +_{\mathfrak{q}} V)^*)^{1/2}) \\ &= \operatorname{dom}(L_{a,\Omega,\Gamma}^{1/2}) = \operatorname{dom}(L_{a,\Omega,\Gamma}^*)^{1/2} = W_\Gamma^{1,2}(\Omega). \end{aligned} \quad (2.19)$$

Proof. For notational simplicity, and without loss of generality, we put the L^∞ -part V_∞ of V equal to zero for the remainder of this proof. Under Hypothesis 2.4, there exists an extension operator \mathcal{E} satisfying

$$(\mathcal{E}f)(x) = f(x) \text{ for a.e. } x \in \Omega, \quad f \in L^2(\Omega), \quad (2.20)$$

with

$$\begin{aligned} \mathcal{E} : W_\Gamma^{1,2}(\Omega) &\rightarrow W^{1,2}(\mathbb{R}^n), \\ \|\mathcal{E}f\|_{W^{1,2}(\mathbb{R}^n)}^2 &\leq C_1 \|f\|_{W^{1,2}(\Omega)}^2, \quad f \in W^{1,2}(\Omega), \end{aligned} \quad (2.21)$$

and

$$\begin{aligned} \mathcal{E} : L^2(\Omega) &\rightarrow L^2(\mathbb{R}^n), \\ \|\mathcal{E}f\|_{L^2(\mathbb{R}^n)}^2 &\leq C_2 \|f\|_{L^2(\Omega)}^2, \quad f \in L^2(\Omega), \end{aligned} \quad (2.22)$$

for some constants $C_j > 0$, $j = 1, 2$ (cf., e.g., [4, Lemma 3.3], [16, Lemma 3.4]).

Let V_{ext} and v_{ext} denote the extensions of V and v , respectively, to all of \mathbb{R}^n defined by setting V_{ext} and v_{ext} identical to zero on $\mathbb{R}^n \setminus \Omega$. Evidently, $V_{\text{ext}} \in L^p(\mathbb{R}^n)$, so there exists a constant $M > 0$ for which (cf., e.g., [18, Lemma 3.7])

$$\begin{aligned} \|v_{\text{ext}} f\|_{L^2(\mathbb{R}^n)}^2 &\leq \varepsilon \|(-\Delta)^{1/2} f\|_{L^2(\mathbb{R}^n)}^2 + M \varepsilon^{-n/(2p-n)} \|f\|_{L^2(\mathbb{R}^n)}^2, \\ f &\in W^{1,2}(\mathbb{R}^n), \quad \varepsilon > 0. \end{aligned} \quad (2.23)$$

Consequently, using (2.20)–(2.23), one estimates

$$\begin{aligned} \|v f\|_{L^2(\Omega)}^2 &= \|v_{\text{ext}} \mathcal{E}f\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq \varepsilon_1 \|(-\Delta)^{1/2} \mathcal{E}f\|_{L^2(\mathbb{R}^n)}^2 + M \varepsilon_1^{-n/(2p-n)} \|\mathcal{E}f\|_{L^2(\mathbb{R}^n)}^2 \\ &= \varepsilon_1 \|\nabla \mathcal{E}f\|_{L^2(\mathbb{R}^n)}^2 + M \varepsilon_1^{-n/(2p-n)} \|\mathcal{E}f\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq \varepsilon_1 C_1 \|\nabla f\|_{L^2(\Omega)}^2 + (\varepsilon_1 C_1 + C_0 M \varepsilon^{-n/(2p-n)}) \|f\|_{L^2(\Omega)}^2 \\ &\leq \varepsilon_1 C_1 \|\nabla f\|_{L^2(\Omega)}^2 + (C_1 + C_0 M) \varepsilon_1^{-n/(2p-n)} \|f\|_{L^2(\Omega)}^2, \\ f &\in W_\Gamma^{1,2}(\Omega), \quad 0 < \varepsilon_1 < 1. \end{aligned} \quad (2.24)$$

To obtain the first term in the second equality above, we applied the 2nd representation theorem (cf., e.g., [22, VI.2.23]) to the non-negative, self-adjoint operator $-\Delta$ on $W^{2,2}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$. The form bound in (2.17) now follows by choosing $\varepsilon = \varepsilon_1 C_1$ throughout (2.24) and noting that

$$\|\nabla f\|_{L^2(\Omega)}^2 = \|(-\Delta_{\Omega,\Gamma})^{1/2} f\|_{L^2(\Omega)}^2, \quad f \in W_\Gamma^{1,2}(\Omega), \quad (2.25)$$

by another application of the 2nd representation theorem (see (2.10) with $a(\cdot) = I_n$), proving item (i).

In view of (2.16) and (2.17), one notes that the hypotheses of [18, Theorem 3.6] are met. The statements in item (ii) thus follow from a direct application of [18, Theorem 3.6]. \square

Remark 2.9. The proof of Theorem 2.8 follows the proof of [18, Theorem 3.12] essentially verbatim with only one notable exception: In [18, Theorem 3.12], Ω is assumed to be a *strongly Lipschitz domain* and hence the Stein extension theorem (cf., e.g., [1, Theorem 5.24] or [26, Theorem 5 in §VI.3.1]) is applied to obtain a total extension operator. In the present case, under the weaker assumptions on Ω , appealing to the Stein extension theorem is *not* permitted; so instead, we apply [4, Lemma 3.3], [16, Lemma 3.4] to obtain the extension operator in (2.20)–(2.22) (cf. [16, Remark 3.5]).

Next, we discuss infinitesimal form boundedness for potential coefficients in the critical exponent case in dimensions $n \geq 3$.

Theorem 2.10. *Assume Hypotheses 2.4 and 2.7. Then V is infinitesimally form bounded with respect to $-\Delta_{\Omega, \Gamma}$,*

$$\| |V|^{1/2} f \|_{L^2(\Omega)}^2 \leq \varepsilon \| (-\Delta_{\Omega, \Gamma})^{1/2} f \|_{L^2(\Omega)}^2 + \eta(\varepsilon) \| f \|_{L^2(\Omega)}^2, \quad f \in W_{\Gamma}^{1,2}(\Omega), \quad \varepsilon > 0. \quad (2.26)$$

As a result, V is infinitesimally form bounded with respect to $L_{a, \Omega, \Gamma}$,

$$\| |V|^{1/2} f \|_{L^2(\Omega)}^2 \leq \varepsilon \operatorname{Re}[\mathbf{q}_{a, \Omega, \Gamma}(f, f)] + \tilde{\eta}(\varepsilon) \| f \|_{L^2(\Omega)}^2, \quad f \in W_{\Gamma}^{1,2}(\Omega), \quad \varepsilon > 0. \quad (2.27)$$

Here η and $\tilde{\eta}$ are non-negative functions defined on $(0, \infty)$, generally depending on Ω , n , and Γ .

Proof. Again, for simplicity, we put the L^∞ -part V_∞ of V equal to zero. The proof is a straightforward modification of the proof of the corresponding result for Lipschitz domains given in [18, Theorem 3.14 (iii)], and we present the modified argument here for completeness. By Sobolev embedding (cf., e.g., [4, Remark 3.4 (ii)]),

$$W_{\Gamma}^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega), \quad 2^* = 2n/(n-2), \quad (2.28)$$

where “ \hookrightarrow ” abbreviates continuous (and dense) embedding, and hence there exists a constant $c > 0$ such that

$$\| f \|_{L^{2^*}(\Omega)}^2 \leq c (\| \nabla f \|_{L^2(\Omega)^n}^2 + \| f \|_{L^2(\Omega)}^2), \quad f \in W_{\Gamma}^{1,2}(\Omega). \quad (2.29)$$

Using Hölder’s inequality, (2.29) implies

$$(f, |W|f)_{L^2(\Omega)} \leq c \| W \|_{L^{n/2}(\Omega)} (\| \nabla f \|_{L^2(\Omega)^n}^2 + \| f \|_{L^2(\Omega)}^2), \quad (2.30)$$

$$f \in W_{\Gamma}^{1,2}(\Omega), \quad W \in L^{n/2}(\Omega).$$

Next, let $\varepsilon > 0$ be given. Since $V \in L^{n/2}(\Omega)$, there exist functions $V_{n/2, \varepsilon} \in L^{n/2}(\Omega)$ and $V_{\infty, \varepsilon} \in L^\infty(\Omega)$ with

$$\| V_{n/2, \varepsilon} \|_{L^{n/2}(\Omega)} \leq \varepsilon/c, \quad V(x) = V_{n/2, \varepsilon}(x) + V_{\infty, \varepsilon}(x) \quad \text{for a.e. } x \in \Omega. \quad (2.31)$$

Applying (2.30) with $W = V_{n/2, \varepsilon}$, one estimates

$$\begin{aligned} \| v f \|_{L^2(\Omega)} &= (f, |V|f)_{L^2(\Omega)} \leq (f, [|V_{n/2, \varepsilon}| + \| V_{\infty, \varepsilon} \|_{L^\infty(\Omega)}] f)_{L^2(\Omega)} \\ &\leq \varepsilon \| \nabla f \|_{L^2(\Omega)^n}^2 + \eta(\varepsilon) \| f \|_{L^2(\Omega)}^2, \quad f \in W_{\Gamma}^{1,2}(\Omega), \end{aligned} \quad (2.32)$$

with

$$\eta(\varepsilon) := \varepsilon + \| V_{\infty, \varepsilon} \|_{L^\infty(\Omega)}. \quad (2.33)$$

Noting the fact that

$$\|\nabla f\|_{L^2(\Omega)^n}^2 = \|(-\Delta_{\Omega,\Gamma})^{1/2} f\|_{L^2(\Omega)}^2, \quad f \in W_\Gamma^{1,2}(\Omega), \quad (2.34)$$

by the 2nd representation theorem (cf., e.g., [22, Theorem VI.2.23]), (2.32) then also yields

$$\|vf\|_{L^2(\Omega)}^2 \leq \varepsilon \|(-\Delta_{\Omega,\Gamma})^{1/2} f\|_{L^2(\Omega)}^2 + \eta(\varepsilon) \|f\|_{L^2(\Omega)}^2, \quad f \in W_\Gamma^{1,2}(\Omega). \quad (2.35)$$

Since $\varepsilon > 0$ was arbitrary and $v = |V|^{1/2}$, (2.26) follows.

To prove (2.27), one notes that the uniform ellipticity condition on a implies

$$\|\nabla f\|_{L^2(\Omega)^n}^2 \leq a_1^{-1} \operatorname{Re}[\mathbf{q}_{a,\Omega,\Gamma}(f, f)], \quad f \in W_\Gamma^{1,2}(\Omega). \quad (2.36)$$

Taking (2.36) together with (2.26) and (2.34), one infers that

$$\|vf\|_{L^2(\Omega)}^2 \leq a_1^{-1} \varepsilon_1 \operatorname{Re}[\mathbf{q}_{a,\Omega,\Gamma}(f, f)] + \eta(\varepsilon_1) \|f\|_{L^2(\Omega)}^2, \quad f \in W_\Gamma^{1,2}(\Omega), \quad \varepsilon_1 > 0. \quad (2.37)$$

The form bound in (2.27) follows by taking $\varepsilon_1 = a_1 \varepsilon$, $\varepsilon > 0$, in (2.37). \square

3. THE CASE OF MATRIX-VALUED DIVERGENCE FORM ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS

In this section we consider uniformly elliptic partial differential operators in divergence form in the vector-valued context, that is, we will focus on $N \times N$ matrix-valued differential expressions \mathbf{L} which act as

$$\mathbf{L}u = - \left(\sum_{j,k=1}^n \partial_j \left(\sum_{\beta=1}^N a_{j,k}^{\alpha,\beta} \partial_k u_\beta \right) \right)_{1 \leq \alpha \leq N}, \quad u = (u_1, \dots, u_N), \quad (3.1)$$

and prove our principal result concerning stability of square root domains with respect to additive perturbations.

To set the stage, we introduce the following set of hypotheses.

Hypothesis 3.1. Fix $n \in \mathbb{N} \setminus \{1\}$, $N \in \mathbb{N}$.

(i) Assume that $\Omega \subset \mathbb{R}^n$ is a non-empty, bounded, open, and connected set satisfying the interior corkscrew condition in Definition 2.1 (i).

(ii) For each $1 \leq \alpha \leq N$, suppose $\Gamma_\alpha \subseteq \partial\Omega$ is a closed subset of $\partial\Omega$ which is either empty or an $(n-1)$ -set, and let $\mathbb{G} = (\Gamma_1, \Gamma_2, \dots, \Gamma_N)$.

(iii) Around every point $x \in \partial\Omega \setminus \bigcap_{\alpha=1}^N \Gamma_\alpha$, suppose there exists an open neighborhood $U_x \subset \mathbb{R}^n$ and a bi-Lipschitz map $\Phi_x : U_x \rightarrow (-1, 1)^n$ such that

$$\Phi_x(x) = 0, \quad (3.2)$$

$$\Phi_x(\Omega \cap U_x) = (-1, 1)^{n-1} \times (-1, 0), \quad (3.3)$$

$$\Phi_x(\partial\Omega \cap U_x) = (-1, 1)^{n-1} \times \{0\}. \quad (3.4)$$

(iv) Define the set

$$\mathcal{W}_\mathbb{G}(\Omega) = \prod_{\alpha=1}^N W_{\Gamma_\alpha}^{1,2}(\Omega), \quad (3.5)$$

where $W_{\Gamma_\alpha}^{1,2}(\Omega)$ is defined as in (2.9) for each $1 \leq \alpha \leq N$, suppose that

$$a_{j,k}^{\alpha,\beta} \in L^\infty(\Omega), \quad 1 \leq j, k \leq n, \quad 1 \leq \alpha, \beta \leq N, \quad (3.6)$$

and assume that the sesquilinear form in $L^2(\Omega)^N$,

$$\begin{aligned} \mathfrak{L}_{a,\Omega,\mathbb{G}}(f, g) &= \sum_{j,k=1}^n \sum_{\alpha,\beta=1}^N \int_{\Omega} d^n x \overline{(\partial_j f_{\alpha})(x)} a_{j,k}^{\alpha,\beta}(x) (\partial_k g_{\beta})(x), \\ f, g &\in \text{dom}(\mathfrak{L}_{a,\Omega,\mathbb{G}}) := \mathcal{W}_{\mathbb{G}}(\Omega), \end{aligned} \quad (3.7)$$

satisfies a uniform ellipticity condition of the form, for some $\lambda > 0$,

$$\text{Re}[\mathfrak{L}_{a,\Omega,\mathbb{G}}(f, g)] \geq \lambda \sum_{\alpha=1}^N \|\nabla f_{\alpha}\|_{L^2(\Omega)^n}^2, \quad f = (f_{\alpha})_{\alpha=1}^N \in \mathcal{W}_{\mathbb{G}}(\Omega). \quad (3.8)$$

We denote by $\mathbf{L}_{a,\Omega,\mathbb{G}}$ the m -sectorial operator in $L^2(\Omega)^N$ uniquely associated to the sesquilinear form $\mathfrak{L}_{a,\Omega,\mathbb{G}}$.

Intuitively, $\mathfrak{L}_{a,\Omega,\mathbb{G}}$ acts on vectors $u = (u_1, u_2, \dots, u_N)$, where each component u_{α} formally satisfies a Dirichlet boundary condition along Γ_{α} and a Neumann condition along the remainder of the boundary, $\partial\Omega \setminus \Gamma_{\alpha}$, $1 \leq \alpha \leq N$ (cf., e.g., [14, Corollary 4.2]).

Hypothesis 3.2. Let $n \in \mathbb{N} \setminus \{1\}$, $N \in \mathbb{N}$, and assume that $\Omega \subset \mathbb{R}^n$ is nonempty and open. Suppose, in addition, that $p > n/2$ and that $V_{\alpha,\beta} \in L^p(\Omega) + L^{\infty}(\Omega)$ for each $1 \leq \alpha, \beta \leq N$.

Assuming Hypotheses 3.1 and 3.2, consider the operator of multiplication by the $N \times N$ matrix-valued function $\mathbf{V} = \{V_{\alpha,\beta}\}_{1 \leq \alpha, \beta \leq N}$ in $L^2(\Omega)^N$ given by

$$(\mathbf{V}f)_{\alpha} = \sum_{\beta=1}^N V_{\alpha,\beta} f_{\beta}, \quad 1 \leq \alpha \leq N, \quad f \in \text{dom}(\mathbf{V}) = \{f \in L^2(\Omega)^N \mid \mathbf{V}f \in L^2(\Omega)^N\}. \quad (3.9)$$

Next, consider the generalized polar decomposition (cf. [20]) for \mathbf{V} :

$$\mathbf{V} = |\mathbf{V}^*|^{1/2} U |\mathbf{V}|^{1/2}, \quad (3.10)$$

where U is an appropriate partial isometry. The sesquilinear form corresponding to \mathbf{V} is then given by

$$\begin{aligned} \mathfrak{V}(f, g) &= (|\mathbf{V}^*|^{1/2} f, U |\mathbf{V}|^{1/2} g)_{L^2(\Omega)^N}, \\ f, g &\in \text{dom}(\mathfrak{V}) = \text{dom}(|\mathbf{V}|^{1/2}) = \text{dom}(|\mathbf{V}^*|^{1/2}). \end{aligned} \quad (3.11)$$

With the $L^p(\Omega)$ assumption on each entry $V_{\alpha,\beta}$, \mathfrak{V} is infinitesimally form bounded with respect to $\mathbf{L}_{a,\Omega,\mathbb{G}}$. In order to prove this, it suffices to consider the case where the L^{∞} -part of each $V_{\alpha,\beta}$ is zero. In this case, one has the estimate

$$|\mathfrak{V}(f, f)|^2 = |(|\mathbf{V}^*|^{1/2} f, U |\mathbf{V}|^{1/2} f)_{L^2(\Omega)^N}|^2 \quad (3.12)$$

$$\leq |(|\mathbf{V}^*|^{1/2} f, |\mathbf{V}|^{1/2} f)_{L^2(\Omega)^N}|^2 \quad (3.13)$$

$$\leq \| |\mathbf{V}^*|^{1/2} f \|_{L^2(\Omega)^N}^2 \| |\mathbf{V}|^{1/2} f \|_{L^2(\Omega)^N}^2 \quad (3.14)$$

$$\begin{aligned} &= \int_{\Omega} (|\mathbf{V}^*(x)|^{1/2} f(x), |\mathbf{V}^*(x)|^{1/2} f(x))_{\mathbb{C}^N} d^n x \\ &\quad \times \int_{\Omega} (|\mathbf{V}(x)|^{1/2} f(x), |\mathbf{V}(x)|^{1/2} f(x))_{\mathbb{C}^N} d^n x \end{aligned} \quad (3.15)$$

$$= \int_{\Omega} (f(x), |\mathbf{V}^*(x)|f(x))_{\mathbb{C}^N} d^n x \int_{\Omega} (f(x), |\mathbf{V}(x)|f(x))_{\mathbb{C}^N} d^n x \quad (3.16)$$

$$\leq \int_{\Omega} \|\mathbf{V}^*(x)\|_2 \|f(x)\|_{\mathbb{C}^N}^2 d^n x \int_{\Omega} \|\mathbf{V}(x)\|_2 \|f(x)\|_{\mathbb{C}^N}^2 d^n x \quad (3.17)$$

$$= \left[\int_{\Omega} \left(\sum_{\alpha=1}^N \sum_{\beta=1}^N |V_{\alpha,\beta}(x)|^2 \right)^{1/2} \|f(x)\|_{\mathbb{C}^N}^2 d^n x \right]^2 \quad (3.18)$$

$$\leq \left[\int_{\Omega} W(x) \|f(x)\|_{\mathbb{C}^N}^2 d^n x \right]^2, \quad f \in \text{dom}(|\mathbf{V}|^{1/2}), \quad (3.19)$$

where we have set

$$W(x) = \sum_{\alpha=1}^N \sum_{\beta=1}^N |V_{\alpha,\beta}(x)| \quad \text{for a.e. } x \in \Omega, \quad (3.20)$$

and used $\|\cdot\|_2$ to denote the Hilbert–Schmidt norm of a matrix in $\mathbb{C}^{N \times N}$. The estimate in (3.19) subsequently implies

$$\begin{aligned} |\mathfrak{V}(f, f)| &\leq \int_{\Omega} \|W(x)^{1/2} f(x)\|_{\mathbb{C}^N}^2 d^n x \\ &= \sum_{\alpha=1}^N \int_{\Omega} |W(x)^{1/2} f_{\alpha}(x)|^2 d^n x \\ &= \sum_{\alpha=1}^N \|W^{1/2} f_{\alpha}\|_{L^2(\Omega)}^2, \quad f \in \text{dom}(|\mathbf{V}|^{1/2}). \end{aligned} \quad (3.21)$$

By hypothesis, one infers that $W \in L^p(\Omega)$. Since $p > n/2$, W is infinitesimally form bounded with respect to $-\Delta_{\Omega, \Gamma_{\alpha}}$ for each $1 \leq \alpha \leq N$ (recalling the notational convention $-\Delta_{\Omega, \Gamma} = L_{I_n, \Omega, \Gamma}$ set forth in (2.13)), with a form bound of the following type:

$$\begin{aligned} \|W^{1/2} f\|_{L^2(\Omega)}^2 &\leq \varepsilon \|(-\Delta_{\Omega, \Gamma_{\alpha}})^{1/2} f\|_{L^2(\Omega)}^2 + M_{\alpha} \varepsilon^{-n/(2p-n)} \|f\|_{L^2(\Omega)}^2, \\ &\quad f \in W_{\Gamma_{\alpha}}^{1,2}(\Omega), \quad 0 < \varepsilon < 1, \quad 1 \leq \alpha \leq N, \end{aligned} \quad (3.22)$$

for appropriate constants $M_{\alpha} > 0$, $1 \leq \alpha \leq N$. Setting $M = \max_{1 \leq \alpha \leq N} M_{\alpha}$ and applying (3.22) to each term of the summation in (3.21), one obtains

$$\begin{aligned} |\mathfrak{V}(f, f)| &\leq \varepsilon \sum_{\alpha=1}^N \|(-\Delta_{\Omega, \Gamma_{\alpha}})^{1/2} f_{\alpha}\|_{L^2(\Omega)}^2 + M \sum_{\alpha=1}^N \varepsilon^{-n/(2p-n)} \|f_{\alpha}\|_{L^2(\Omega)}^2 \\ &= \varepsilon \sum_{\alpha=1}^N \|\nabla f_{\alpha}\|_{L^2(\Omega)}^2 + M \varepsilon^{-n/(2p-n)} \|f\|_{L^2(\Omega)^N}^2, \\ &\quad f \in \mathcal{W}_{\mathbb{G}}(\Omega), \quad 0 < \varepsilon < 1. \end{aligned} \quad (3.23)$$

Finally, applying the uniform ellipticity condition (3.8) to (3.23), one obtains the form bound,

$$\begin{aligned} |\mathfrak{V}(f, f)| &\leq \lambda^{-1} \varepsilon \text{Re}[\mathfrak{L}_{a, \Omega, \mathbb{G}}(f, f)] + M \varepsilon^{-n/(2p-n)} \|f\|_{L^2(\Omega)^N}^2, \\ &\quad f \in \mathcal{W}_{\mathbb{G}}(\Omega), \quad 0 < \varepsilon < 1. \end{aligned} \quad (3.24)$$

By suitably rescaling ε throughout (3.24), one infers that \mathbf{V} is infinitesimally form bounded with respect to $\mathbf{L}_{a,\Omega,\mathbb{G}}$. Infinitesimal form boundedness of \mathbf{V} with respect to $\mathbf{L}_{a,\Omega,\mathbb{G}}$ is summarized in the following result.

Theorem 3.3. *Assume Hypotheses 3.1 and 3.2. Then \mathbf{V} is infinitesimally form bounded with respect to $\mathbf{L}_{a,\Omega,\mathbb{G}}$ and there exist constants $M > 0$ and $\varepsilon_0 > 0$ such that*

$$|\mathfrak{V}(f, f)| \leq \varepsilon \operatorname{Re}[\mathfrak{L}_{a,\Omega,\mathbb{G}}(f, f)] + M\varepsilon^{-n/(2p-n)} \|f\|_{L^2(\Omega)^N}^2, \quad f \in \mathcal{W}_{\mathbb{G}}(\Omega), \quad 0 < \varepsilon < \varepsilon_0. \quad (3.25)$$

In view of Theorem 3.3, the form sum $\mathbf{L}_{a,\Omega,\mathbb{G}} +_{\mathfrak{q}} \mathbf{V}$ is well-defined and represents an m-sectorial operator in $L^2(\Omega)^N$. Our next result extends stability of square root domains to $\mathbf{L}_{a,\Omega,\mathbb{G}}$ and $\mathbf{L}_{a,\Omega,\mathbb{G}} +_{\mathfrak{q}} \mathbf{V}$.

Theorem 3.4. *Assume Hypotheses 3.1 and 3.2 and let \mathbf{V} denote the operator of component-wise multiplication in $L^2(\Omega)^N$ defined in (3.9). Then*

$$\begin{aligned} \operatorname{dom}((\mathbf{L}_{a,\Omega,\mathbb{G}} +_{\mathfrak{q}} \mathbf{V})^{1/2}) &= \operatorname{dom}((\mathbf{L}_{a,\Omega,\mathbb{G}} +_{\mathfrak{q}} \mathbf{V})^*)^{1/2} \\ &= \operatorname{dom}(\mathbf{L}_{a,\Omega,\mathbb{G}}^{1/2}) = \operatorname{dom}((\mathbf{L}_{a,\Omega,\mathbb{G}}^*)^{1/2}) = \mathcal{W}_{\mathbb{G}}(\Omega). \end{aligned} \quad (3.26)$$

Proof. Let \mathbf{V} be factored into the form $\mathbf{V} = \mathbf{B}^* \mathbf{A}$

$$\mathbf{A} = U|\mathbf{V}|^{1/2}, \quad \mathbf{B} = |\mathbf{V}|^{1/2}, \quad \operatorname{dom}(\mathbf{A}) = \operatorname{dom}(\mathbf{B}) = \operatorname{dom}(|\mathbf{V}|^{1/2}), \quad (3.27)$$

according to the generalized polar decomposition in (3.10). One observes that $\mathcal{W}_{\mathbb{G}}(\Omega) \subset \operatorname{dom}(\mathbf{A}) = \operatorname{dom}(\mathbf{B})$. Therefore, [14, Theorem 9.2] implies

$$\operatorname{dom}(\mathbf{A}) \supseteq \operatorname{dom}(\mathbf{L}_{a,\Omega,\mathbb{G}}^{1/2}), \quad \operatorname{dom}(\mathbf{B}) \supseteq \operatorname{dom}((\mathbf{L}_{a,\Omega,\mathbb{G}}^*)^{1/2}). \quad (3.28)$$

Next, let $\mathbf{D}_{\Omega,\mathbb{G}}$ denote the non-negative self-adjoint operator uniquely associated to the sesquilinear form

$$\begin{aligned} \mathfrak{D}_{\Omega,\mathbb{G}}(f, g) &= \sum_{j,k=1}^n \sum_{\alpha,\beta=1}^N \int_{\Omega} \overline{(\partial_j f_{\alpha})(x)} \delta_{j,k} \delta_{\alpha,\beta} (\partial_k g_{\beta})(x) d^n x, \\ f, g &\in \operatorname{dom}(\mathfrak{D}_{\Omega,\mathbb{G}}) := \mathcal{W}_{\mathbb{G}}(\Omega), \end{aligned} \quad (3.29)$$

where $\delta_{j,k}$ denotes the Kronecker delta symbol. (One notes that (3.29) is simply (3.7) with tensor coefficients $a_{j,k}^{\alpha,\beta} = \delta_{j,k} \delta_{\alpha,\beta}$, $1 \leq j, k \leq n$, $1 \leq \alpha, \beta \leq N$.) Then (cf., e.g., the discussion preceding [14, Theorem 9.2])

$$(\mathbf{D}_{\Omega,\mathbb{G}} f)_{\alpha} = -\Delta_{\Omega,\Gamma_{\alpha}} f_{\alpha}, \quad 1 \leq \alpha \leq N, \quad f \in \operatorname{dom}(\mathbf{D}_{\Omega,\mathbb{G}}) = \prod_{\beta=1}^N \operatorname{dom}(-\Delta_{\Omega,\Gamma_{\beta}}), \quad (3.30)$$

where we have used the notational convention $-\Delta_{\Omega,\Gamma_{\alpha}} = L_{I_n, \Omega, \Gamma_{\alpha}}$ set forth in (2.13). In addition (cf. the discussion in the proof to [14, Theorem 9.2]),

$$(\mathbf{D}_{\Omega,\mathbb{G}}^{1/2} f)_{\alpha} = (-\Delta_{\Omega,\Gamma_{\alpha}})^{1/2} f_{\alpha}, \quad 1 \leq \alpha \leq N, \quad f \in \operatorname{dom}(\mathbf{D}_{\Omega,\mathbb{G}}^{1/2}) = \mathcal{W}_{\mathbb{G}}(\Omega). \quad (3.31)$$

As a result of (3.31), the bound in (3.23) actually implies

$$\begin{aligned} \|\mathbf{A}f\|_{L^2(\Omega)^N}^2 &\leq \varepsilon \|\mathbf{D}_{\Omega,\mathbb{G}}^{1/2} f\|_{L^2(\Omega)^N}^2 + M\varepsilon^{-n/(2p-n)} \|f\|_{L^2(\Omega)^N}^2, \\ f &\in \mathcal{W}_{\mathbb{G}}(\Omega), \quad 0 < \varepsilon < 1. \end{aligned} \quad (3.32)$$

Hence, [18, Lemma 2.12] guarantees the existence of constants $M_1 > 0$, $q > 0$, and $E_0 \geq 1$ such that

$$\|\mathbf{A}(\mathbf{D}_{\Omega, \mathbb{G}} + EI_{L^2(\Omega)^N})^{-1/2}\|_{\mathcal{B}(L^2(\Omega)^N)} \leq M_1 E^{-q}, \quad E > E_0. \quad (3.33)$$

Since $\text{dom}(\mathbf{L}_{a, \Omega, \mathbb{G}}^{1/2}) = \text{dom}(\mathbf{D}_{\Omega, \mathbb{G}}^{1/2})$, [18, Lemma 2.11] yields the existence of a constant $C > 0$ such that

$$\sup_{E \geq 1} \|(\mathbf{L}_{a, \Omega, \mathbb{G}} + EI_{L^2(\Omega)^N})^{1/2}(\mathbf{D}_{\Omega, \mathbb{G}} + EI_{L^2(\Omega)^N})^{-1/2}\|_{\mathcal{B}(L^2(\Omega)^N)} \leq C. \quad (3.34)$$

The estimates in (3.33) and (3.34) imply

$$\|\mathbf{A}(\mathbf{L}_{a, \Omega, \mathbb{G}} + EI_{L^2(\Omega)^N})^{-1/2}\|_{\mathcal{B}(L^2(\Omega)^N)} \leq \widehat{M}_1 E^{-q}, \quad E > E_0, \quad (3.35)$$

for an appropriate constant $\widehat{M}_1 > 0$. A similar argument involving adjoints can be used to show

$$\|\overline{(\mathbf{L}_{a, \Omega, \mathbb{G}} + EI_{L^2(\Omega)^N})^{-1/2} \mathbf{B}^*}\|_{\mathcal{B}(L^2(\Omega)^N)} \leq \widehat{M}_2 E^{-q}, \quad E > E_0, \quad (3.36)$$

for an appropriate constant $\widehat{M}_2 > 0$.

Finally, in light of (3.28), (3.35), (3.36), and the fact that (cf. [14, Theorem 9.2])

$$\text{dom}(\mathbf{L}_{a, \Omega, \mathbb{G}}^{1/2}) = \text{dom}((\mathbf{L}_{a, \Omega, \mathbb{G}}^*)^{1/2}), \quad (3.37)$$

the string of equalities in (3.26) follows from an application of [18, Corollary 2.7]. We note that [18, Hypothesis 2.1 (iii)] holds in the present setting since

$$\begin{aligned} & \|\overline{\mathbf{A}(\mathbf{L}_{a, \Omega, \mathbb{G}} + EI_{L^2(\Omega)^N})^{-1} \mathbf{B}^*}\|_{\mathcal{B}(L^2(\Omega)^N)} \\ & \leq \|\mathbf{A}(\mathbf{L}_{a, \Omega, \mathbb{G}} + EI_{L^2(\Omega)^N})^{-1/2}\|_{\mathcal{B}(L^2(\Omega)^N)} \\ & \quad \times \|\overline{(\mathbf{L}_{a, \Omega, \mathbb{G}} + EI_{L^2(\Omega)^N})^{-1/2} \mathbf{B}^*}\|_{\mathcal{B}(L^2(\Omega)^N)}, \quad E > 0, \end{aligned} \quad (3.38)$$

and the estimates in (3.35), (3.36) yield decay to zero as $E \uparrow \infty$ of the factors on the right-hand side of (3.38). \square

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